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Initial-value problems for Rossby waves in a shear flow with critical level

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The time-dependent evolution of sheared Rossby waves starting from an initial disturbance is studied for the simple case in which the shear is uniform. The uniform-shear assumption allows explicit solutions to be obtained which are useful in addressing the issue of the long-time asymptotic approach to normal modes and in assessing the relative importance of viscosity, nonlinearity and time-dependence in the evolution of Rossby waves in the presence of critical layers.

1. Introduction

In a shear flow $U(y)$, a wave with a well-defined phase speed c will encounter a critical level where $U(y)-c$ vanishes. What happens to a wave as it propagates to such a location is of both theoretical and practical importance (see Tung 1979). It would seem that the most natural way to answer the question concerning the fate of the wave is to release in the shear flow an initial wave disturbance with the required characteristics and study the evolution of the wave in time. However, this is not the procedure adopted in most past studies, presumably because of the difficulties, both analytic and numerical, involved in treating the long-time evolution of free oscillations under shear, as mentioned by Stewartson (1978). Our present intuitions on such problems as 'absorption' or 'reflection' of waves approaching a critical level have mostly been gained instead either from normal-mode/steady-state theory or from asymptotic/numerical solutions of *continuously forced* problems. These intuitions, as we shall see, are not always applicable to the free-wave problem at hand.

In normal-mode theory, the solution, say the stream function ψ, is assumed to be of the following separable form:†

$$\psi(x,y,t) = e^{ik(x-ct)}\,\phi(y), \tag{1.1}$$

where k is the wavenumber and c is the phase speed, both in the x-direction, along which $U(y)$ flows. Substituting (1.1) into the (linear inviscid) vorticity equation, one then obtains the usual Rayleigh–Kuo equation for the modal structure $\phi(y)$:

$$\left(\frac{d^2}{dy^2}-k^2\right)\phi+\frac{\beta-U_{yy}}{U(y)-c}\phi = 0. \tag{1.2}$$

A *local* Frobenius expansion near $y = y_c \equiv c/U_y$ reveals that $\phi(y)$ has a logarithmic singularity at y_c, though it is often not possible to determine if y_c is real based simply on such a local expansion. A misconception is often attached to this normal-mode solution: It is sometimes assumed, albeit implicitly, that any disturbance in a shear flow will eventually develop a critical-level singularity if at least part of the spectrum

† The phrase 'normal mode' is used in this paper loosely to denote 'solution in the modal form' (1.1).

of phase speeds lies within the range of $U(y)$. And, with the balance of terms implied by the presence of this singularity, various continuation conditions are obtained across the critical layer which supposedly determine the fate of the 'incident wave'.

It was pointed out by Lin (1945), based on an asymptotic solution of the Orr–Sommerfeld equation in the limit of large Reynolds numbers (i.e. small viscosity), that the solution to (1.2) with $c_i > 0$, the so-called self-excited disturbance, does not need an 'inner friction layer'; the solution on one side of $U - c_r$ can be analytically continued to the other side.† The situation for the neutral ($c_i = 0$) and damped ($c_i < 0$) disturbances, however, is entirely different. The effect of viscosity was found by Lin (1945) to be important for this case, and one or more 'inner friction layers' must be introduced near the critical level. The resulting continuation scheme was found to be different from that obtained from analytic continuation.

Benney & Bergeron (1969) suggested a different continuation scheme based on the dominance of nonlinearity in the critical layer. Using this nonlinear critical-layer continuation condition, they obtained a new class of *neutral* normal-mode solutions. Haberman (1972) later extended the theory to the case where both nonlinearity and viscosity are important in the critical layer. The presence of a critical-level singularity in the linear inviscid solution (to (1.2) for example) is an essential ingredient in these nonlinear continuation schemes. The $1/(y-y_c)$ type singularity for the wave vorticity (or $d^2\phi/dy^2$) indicates that, even if the wave's amplitude is small away from the critical level, it will become large (and hence nonlinear) sufficiently close to the critical level, assuming, that is, that such a singularity occurs on the real axis. That the eigenvalue solutions to (1.2) with the nonlinear continuation condition of Benney & Bergeron (1969) are neutral is, at least, consistent with such an assumption.

The interdependence of the eigenvalue c (in particular, c_i) and the scaling for the continuation schemes appears paradoxical: if the eigensolution to (1.2) has a large c_i then the fact that the singularity occurs away from the real axis would seem to invalidate to some degree the nonlinear scalings used near the critical level. On the other hand, the eigenvalue problem for c is often not completely defined until a continuation scheme is specified to connect the solution on one side of $y - y_c = 0$ to the other side. A simple analytic continuation of the linear inviscid solution may give a mathematical solution, but this may not always be the correct one physically, as the $c_i < 0$ case considered by Lin (1945) amply demonstrates.‡

To resolve this dilemma, it is desirable that

(i) an asymptotic solution to the nonlinear equation be obtained that solves the (global) eigenvalue problem, or

(ii) an initial-value problem be studied to trace the evolution of a disturbance into the nonlinear regime.

Because of the mathematical difficulties involved, existing analytic results on nonlinear critical layers are only locally valid and have to be 'matched' to the outer solutions of the assumed Frobenius form. No result analogous to the uniformly valid asymptotic solutions of Lin (1957) (which is for the viscous critical layer) is currently available.

† Note that this is an asymptotic result in the limit of vanishing viscosity. For the case of small but finite viscosity, the effect of viscosity may have to be included in the critical layer for weakly unstable waves.

‡ The eigensolutions to (1.2) that one would obtain using analytic continuation would come in complex-conjugate pairs, and hence the unphysical result that there is no profile that is stable to infinitesimal normal-mode perturbations because associated with each stable eigenvalue ($c_i < 0$) there would be an unstable one ($c_i > 0$).

The second approach, that of an initial-value problem, will be adopted in the present paper for a stable (Couette) profile, for which the problem of viscous *vs.* nonlinear continuation schemes arises. Our result seems to indicate that the nonlinear scaling based on the normal-mode solutions is probably not applicable to the free problem at hand, though we will show in a second paper that it is applicable to the forced problem when forcing is continuously applied over a sufficiently long time period (of the order of the reciprocal of the forcing amplitude).

It will first be shown that an initially linear disturbance in a stable shear flow does not develop singularities in its flow field for all time. In other words, the 'normal-mode' solution from (1.1) and (1.2) is *not* the steady-state limit of any physically reasonable initial disturbance. We have managed to show, however, that it *is* the steady-state limit of an initial condition with a delta function singularity in its vorticity field located exactly at the critical level. The time-dependent linear solution together with the nonlinear correction is calculated and the effect of nonlinearity is found to be uniformly small for all time. In other words, there is no region in the flow domain (including the critical level) where nonlinearity plays a significant role if the initial wave amplitude is asymptotically small.

Although as a mathematical representation, an infinite sum and/or integral of all normal modes, however continued through their respective singularities in the complex plane, can be used to represent any solution of the initial-value problem (as long as these modes form a complete set), such a representation may not always be convenient. Often the behaviour of the solution can be misleadingly interpreted from its individual normal-mode components. In particular, it is shown that the scaling of various terms (e.g. viscous and nonlinear terms) inside a critical layer based on the modal structure inferred from (1.2) may not be applicable to free-wave problems, although it has been shown to apply to problems with continuous forcing (Dickinson 1970; Warn & Warn 1976, 1978; Béland 1976; Stewartson 1978; Brown & Stewartson 1978).

Some of the issues mentioned above are addressed here in a model that assumes that the pre-existing zonal flow is of uniform shear, i.e.

$$U = U'(y-y_0). \tag{1.3}$$

(The shear $U'(t)$ and the location of zero-wind line $y_0(t)$ are allowed to depend on t, though this feature is not emphasized in the present paper to avoid distraction from the main issues.)

The restriction to the uniform-shear case is admittedly made for analytic convenience. However, the monotonic nature of the flow (1.3) has the desired feature that, for any real c, there is always a $U(y)$ that can match it to give a critical level. This is suitable for our purpose of studying critical level behaviour, but is not suitable for studying noncritical waves. It should also be pointed out that (1.3) is a stable profile. From a WKB sense the solution qualitatively typifies the class of stable profiles for which $\beta - U_{yy}$ is always positive. The solution for profiles for which $\beta - U_{yy}$ changes sign in the domain is expected to be qualitatively different. Some comments on this unstable case will be made in §7.

Using the convected coordinate formulation of Phillips (1966), Hartman (1975) and Yamagata (1976*a*, *b*) to take advantage of the assumption in (1.3), we have obtained the solution for the viscous initial-value problem along with the nonlinear corrections. This solution is used to show that nonlinearity will not play a dominant role in the evolution of a free disturbance if the initial disturbance is not already nonlinear. The evolution of the wave is dominated by the kinematic shearing of its meridional wavenumbers and by dissipation. For the case where the initial wave spectrum has well-defined wavenumbers, the waves as a group eventually move southward towards a so-called stagnation level, determined by the location where $U(y)-c=0$, with c being the barotropic wave speed in the absence of shear. Even though the initial disturbances that constitute the wave packet in general do not have a well-defined phase speed, it is shown that, during the later stages of its evolution, the wave packet as a whole behaves, from kinematic considerations, as if there exists a well-defined phase speed c (the so-called nominal phase speed of the packet). Because of this kinematic feature, some of the normal-mode results concerning critical-level absorptions can be reinterpreted in the initial-value problem treated here, even though no solution of the modal form exists and even when nonlinearity is taken into account.†

Viscosity acts continuously on the waves depending on the age of the wave since generation. There is no 'singular absorption' of wave energy associated with critical levels, as would be inferred from local expansions of the normal-mode equation (e.g. Booker & Bretherton 1967; Dickinson 1969). Nevertheless, an interpretation of the critical level absorption phenomenon can be made in the limit of infinite time and small (but not zero) viscosity. This is based on the observation that, since the wave spends most of its time near the stagnation level (Bretherton 1966), viscosity would have enough time to act on the wave near this location, while the effect of dissipation on the wave *en route* is negligibly small. This argument still does not give the stagnation level as the primary location where the wave's energy is 'absorbed'. Indeed, it can be shown that the wave energy density has decreased so much due to the time-dependent wave–mean interaction in the wave's southward journey that there is often not much left to be 'absorbed' at its final destination. A detailed examination of the wave–mean interaction problem is given in §7.

2. Problem formulation

The following simple barotropic vorticity equation shall be taken as the model equation for studying the time-dependent evolution of Rossby waves in a sheared zonal flow $U(y,t)$:

$$\left(\frac{\partial}{\partial t}+U(y,t)\frac{\partial}{\partial x}\right)\zeta+\epsilon \mathrm{J}(\psi,\zeta)+(\beta-U_{yy})\frac{\partial}{\partial x}\psi=\nu\nabla^2\zeta, \tag{2.1}$$

where $\epsilon\psi$ is the perturbation stream function,

$$\zeta=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi\equiv\nabla^2\psi$$

is the perturbation vorticity multiplied by the factor $1/\epsilon$,

$$\mathrm{J}(A,B)\equiv A_xB_y-A_yB_x$$

is the nonlinear Jacobian operator, and ν is the eddy-viscosity coefficient. Equation (2.1) is to be solved subject to the initial condition

$$\zeta(x,y,t=0)=\zeta_0(x,y). \tag{2.2}$$

† It should be noted that the commonly accepted notion of absorption at the critical level was deduced either from a steady modal solution using the viscous continuation of Lin (1945) (e.g. Dickinson 1969), or from an initial-value calculation for forced waves neglecting nonlinearity (e.g. Dickinson 1970; Booker & Bretherton 1967).

The boundary conditions are

$$\zeta = 0 \quad (y = \pm L), \tag{2.3}$$

$$\zeta \text{ periodic in } x \text{ with period of } 2\pi a \cos\theta_0, \tag{2.4a}$$

the length of the zonal circle at latitude $\theta = \theta_0$. For localized disturbances, a more convenient condition may be

$$\zeta \to 0 \quad \text{as} \quad x \to \pm\infty. \tag{2.4b}$$

For the meridional boundary condition in (2.3), two cases can be considered within the present formulation. In a finite β-plane approximation, L is taken as $\frac{1}{2}\pi a$. In an infinite β-plane, $L \to \infty$. This latter case corresponds to the interpretation that y is the Mercator coordinate defined by $\tanh(y/a) = \sin\theta$. (For plane waves, the condition $\zeta = 0$ at $y \to \pm\infty$ will be relaxed to the boundedness condition at infinities.)

The problem of interest here is the one in which ϵ, a measure of the *initial* amplitude of the disturbance, is small, i.e. $\epsilon \ll 1$. Therefore it seems reasonable to assume that *initially* the problem is linear. Whether or not nonlinearity will eventually become important as time increases is a question that cannot be answered by simple arguments, especially for the present case with possible critical levels. It will be shown in later sections that *unforced* waves in a stable mean flow will never become nonlinear if they are not so initially. In §3 the linear equation, obtained from (2.1) by setting $\epsilon = 0$, will be solved first.

3. The linear viscous problem

We solve in this section the linear version of (2.1) for the case of a uniform shear $\partial U/\partial y \equiv U'$. The governing equation reduces to

$$\left(\frac{\partial}{\partial t} + U(y,t)\frac{\partial}{\partial x}\right)\zeta + \beta\frac{\partial}{\partial x}\psi - \nu\nabla^2\zeta = 0, \tag{3.1}$$

with $\zeta = \nabla^2\psi$. The general initial condition is written as

$$\zeta(x,y,0) = \zeta_0 = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta_0(k,l)\,\mathrm{e}^{\mathrm{i}kx+\mathrm{i}ly}\,\mathrm{d}k\,\mathrm{d}l, \tag{3.2}$$

where it is understood that the integral transform in k is to be replaced by a Fourier series if the solution is to be periodic in x (2.4a). Also, the integral transform in l is to be replaced by a Fourier sine series if the domain is finite in the y-direction. These changes are discussed in more detail in appendix A. Here we will continue to use the integral sign in a symbolic sense.

For the present case where the shear U' is independent of y, it is convenient to transform the coordinate system into one moving with the mean flow U by defining

$$\xi = x - \int_0^t U\,\mathrm{d}t, \quad \eta = y,$$

$$\tau = t, \quad T(\tau) \equiv \int_0^\tau \frac{\partial}{\partial y}U\,\mathrm{d}t.$$

These have been called the 'convected coordinates' by Phillips (1966) and Hartman (1975). Noting that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \eta} - T(\tau)\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} = \frac{\partial}{\partial \tau},$$

(3.1) becomes

$$\frac{\partial}{\partial \tau}\zeta+\beta\frac{\partial}{\partial \xi}\psi-\nu\left[\frac{\partial^2}{\partial \xi^2}+\left(\frac{\partial}{\partial \eta}-T(\tau)\frac{\partial}{\partial \xi}\right)^2\right]\zeta = 0, \tag{3.3}$$

where now

$$\zeta=\left[\frac{\partial^2}{\partial \xi^2}+\left(\frac{\partial}{\partial \eta}-T(\tau)\frac{\partial}{\partial \xi}\right)^2\right]\psi. \tag{3.4}$$

Note that (3.3) is separable in ξ and η, while the original (3.1) is not separable in the space coordinates because of the non-uniform U. The initial condition (3.2) can be rewritten as

$$\zeta_0(\xi,\eta)=\frac{1}{(2\pi)^2}\iint \hat{\zeta}_0(k,l)\,e^{ik\xi+il\eta}\,dk\,dl, \tag{3.5}$$

there being no difference between x and ξ at $t=\tau=0$.

To satisfy this general initial condition, the solution is first written in the form

$$\zeta=\frac{1}{(2\pi)^2}\iint e^{ik\xi+il\eta}\,\hat{\zeta}(k,l,\tau)\,dk\,dl, \tag{3.6}$$

with

$$\hat{\zeta}(k,l,0)=\hat{\zeta}_0(k,l).$$

For $\tau>0$, the equation for $\hat{\zeta}(k,l,\tau)$ is found from (3.3) and (3.4) to be

$$\frac{d}{d\tau}\hat{\zeta}+i\beta k[k^2+(l-T(\tau)k)^2]^{-1}\hat{\zeta}+\nu[k^2+(l-T(\tau)k)^2]\hat{\zeta}=0. \tag{3.7}$$

This being a first-order ordinary differential equation in τ, an exact solution can be found. The solution is

$$\hat{\zeta}(k,l,\tau)=\hat{\zeta}_0(k,l)\exp\{i\Omega(k,l,\tau)-D(k,l,\tau)\}, \tag{3.8}$$

where

$$\Omega(k,l,\tau)\equiv-\int_0^\tau\frac{\beta k\,d\tau}{k^2+(l-T(\tau)k)^2} \tag{3.9}$$

$$D(k,l,\tau)\equiv\nu\int_0^\tau[k^2+(l-T(\tau)k)^2]\,d\tau. \tag{3.10}$$

For the subcase where the shear is independent of time,

$$T(\tau)\equiv\int_0^\tau U'\,dt=U'\tau,$$

$$\Omega=\frac{\beta}{U'k}\left[\tan^{-1}\left(U'\tau-\frac{l}{k}\right)+\tan^{-1}\frac{l}{k}\right], \tag{3.11}$$

$$D=\frac{\nu k^2}{U'}\left\{\left[\left(U'\tau-\frac{l}{k}\right)+\left(\frac{l}{k}\right)\right]+\frac{1}{3}\left[\left(U'\tau-\frac{l}{k}\right)^3+\left(\frac{l}{k}\right)^3\right]\right\}. \tag{3.12}$$

The corresponding solution for the nonrotating case has previously been obtained by Hartman (1973). The solution (3.6) with $\hat{\zeta}(k,l,\tau)$ thus obtained is valid in an infinite domain, satisfying (2.3) with $L\to\infty$ and (2.4*b*). The modification for the finite-domain cases is discussed in appendix A.

Before we proceed to discuss the solution in more detail for various specific initial conditions, we will first show that nonlinearity will not become important as time increases for the general case.

4. The nonlinear correction

To show that nonlinearity will not play an important role *anywhere* in the flow domain, we need to demonstrate that

(i) the linear solution remains order one everywhere in the domain for all time even in the absence of viscosity, and that

(ii) the nonlinear correction to the linear solution for the nonlinear equation (2.1) remains everywhere order ϵ for all time.

From the approach of asymptotic analysis for small ϵ, the abovementioned tasks are equivalent to proving that (i) the linear solution obtained in §3 as the leading-order solution for small ϵ is *uniformly* bounded for all space and all time, and (ii) the order-ϵ asymptotic solution to (2.1), when divided by ϵ, is also uniformly bounded for all space and time. These proofs are rather tedious and so will be relegated to appendix B. Here the principal conclusion is stated: *for Rossby waves evolving from bounded initial vorticity disturbances, the effect of nonlinearity will not become significant if it is not so initially.* The nonlinearity in the initial condition is measured by the parameter ϵ, which can be taken to be the ratio between the initial magnitude of the perturbation velocity and the value of the basic zonal flow at the location of the maximum of the initial disturbance.

It should be pointed out that, like most asymptotic analyses, the result quoted above has only been shown to be valid, strictly speaking, for vanishingly small ϵ. In real applications, nonlinearity may make a quantitative difference to the solution when ϵ is not very small. However, as discussed in appendix B and also in §7, the dominant effect of nonlinearity will first show up near turning points, rather than near critical levels.

5. An exact solution and interpretation

It has been shown in appendix B, as can also be verified easily, that an exact solution to the nonlinear viscous equation (2.1) is

$$\zeta(x,y,t) = \mathrm{Re}\{a\,\mathrm{e}^{-D(k,l,t)}\,\mathrm{e}^{\mathrm{i}\Theta}\},$$

where

$$\Theta \equiv k\left(x-\int_0^t U(y,t')\,\mathrm{d}t'\right)+ly+\Omega(k,l,t). \tag{5.1}$$

For the case where the shear U' is independent of time (as well as space), (5.1) can be put into the following more explicit form (Yamagata (1976*a*, *b*) previously obtained a similar solution for the linear inviscid problem):

$$\zeta(x,y,t) = \mathrm{Re}\left\{a\exp\left\{-\left(\frac{\nu k^2}{U'}\right)\left[U't+\frac{1}{3}\left(U't-\frac{l}{k}\right)^3+\left(\frac{l}{k}\right)^3\right]\right\}\mathrm{e}^{\mathrm{i}\Theta}\right\}, \tag{5.2}$$

with now

$$\Theta = k(x-U'yt)+ly+\frac{\beta}{kU'}\left[\tan^{-1}\left(U't-\frac{l}{k}\right)+\tan^{-1}\frac{l}{k}\right].$$

This simple solution will be used here to illustrate several concepts which appear to be also relevant to the more general cases.

(i) As has been pointed out for the non-rotating case by Orr (1907); Rosen (1971) and Hartman (1973), the exact solution in the presence of shear is *non-separable*, with the factor

$$\mathrm{e}^{-\mathrm{i}kU'yt},$$

even though (3.1) is a separable equation. This fact implies that a single normal-mode solution of the form

$$e^{ik(x-ct)}\,\phi(y)$$

cannot be the required solution to the initial-value problem. (See §6 for the exceptional case.)

(ii) The solution is *non-singular*, even as $t \to \infty$. The viscous solution (5.2) decays in time and approaches the trivial solution as $t \to \infty$. The *inviscid* vorticity distribution remains bounded for all time and approaches the configuration of a plane wave with a constant amplitude in a coordinate system moving with the mean sheared flow, i.e.

$$\lim_{\substack{\nu \to 0 \\ t \to \infty}} \zeta = a \cos\left\{k\xi + l\eta + \frac{\beta}{kU'}\left[\tfrac{1}{2}\pi + \tan^{-1}\frac{l}{k}\right]\right\}. \tag{5.3}$$

Equation (5.3), obtained by letting $t \to \infty$ in the initial-value problem, is absent from the set of steady-state normal-mode solutions obtained by solving (3.1) with $\partial/\partial t = 0$ or $\partial/\partial t = -c\partial/\partial x$. On the other hand, the steady-state (trivial) solution in the presence of viscosity is correctly obtained from the normal-mode theory for the case of vanishingly small viscosity, *provided* that the viscous continuation of Lin (1957) is used across the critical layer. This condition gives a negative c_i for the present stable case and hence yields an exponential decay in time. Incidentally, it is thus seen that the solution in the limit $t \to \infty$ and then $\nu \to 0$, is different from the limit $\nu \to 0$ and then $t \to \infty$. In an initial-value approach, this ambiguity becomes academic for finite times.

(iii) The factor

$$e^{i\Omega(k,\,l,\,t)}$$

in (5.2) corresponds to the frequency factor

$$e^{i\omega t}$$

in normal-mode theory. The quantity Ω can actually be written as

$$\Omega(k, l, t) = \int_0^t \omega(k, l, t')\,dt', \tag{5.4}$$

where

$$\omega(k, l, t) \equiv \frac{\beta k}{k^2 + (l - kT(t))^2} \tag{5.5}$$

is recognized as the intrinsic Rossby-wave frequency if the meridional wavenumber of the wave in fixed coordinates (x, y) is identified with $l - kT(t)$ (see also Yamagata 1976*a*). In the absence of shear, ω becomes a constant, and (5.4) is simply

$$\Omega = \omega t.$$

One recovers the normal-mode frequency factor.

(iv) One easily obtains from (5.1) the stream function

$$\psi = -\frac{\zeta(x, y, t)}{k^2 + (l - kT(t))^2}. \tag{5.6}$$

The kinetic-energy density of the wave is calculated as

$$\begin{aligned} K(t) &\equiv \tfrac{1}{2}\{\psi_x^2 + \psi_y^2\} \\ &= \frac{\tfrac{1}{2}a^2 \exp\{-2D\} \sin^2\theta}{k^2 + (l - kT(t))^2}. \end{aligned} \tag{5.7}$$

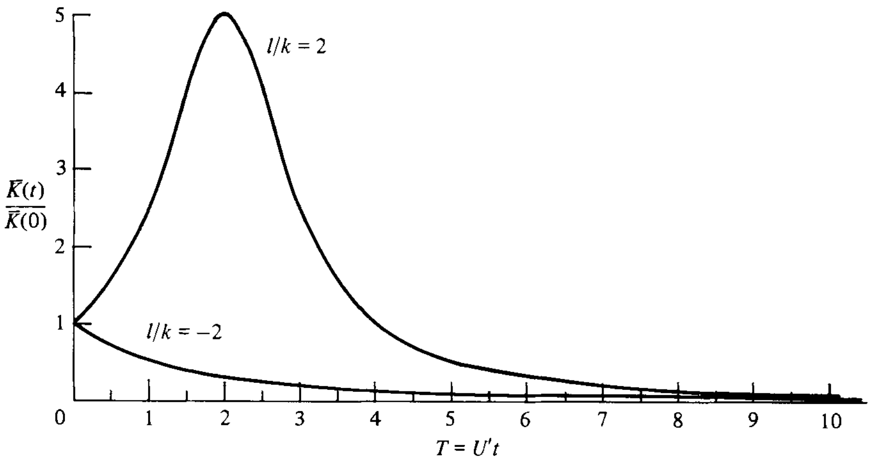

FIGURE 1. Averaged wave kinetic-energy density $\overline{K}(t)$ *vs.* normalized time $T(t) = U't$. $U't = 1$ represents approximately one day for a typical shear of 1 m/s per degree latitude.

And so the zonally averaged value is

$$\overline{K}(t) = \frac{\frac{1}{4}a^2 \, e^{-2D}}{k^2 + (l - kT(t))^2}. \tag{5.8}$$

In figure 1 we depict the ratio

$$\frac{\overline{K}(t)}{\overline{K}(0)} = \frac{1 + (l/k)^2}{1 + (l/k - T(t))^2} \tag{5.9}$$

for the inviscid case. It is seen from the figure that for the wave with $l/k > 0$ (i.e. waves whose initial phase tilts northwest to southeast) there is an initial transient growth in energy density, reaching a maximum ratio of $1 + (l/k)^2$, which can be quite large in some cases. (For example, the amplification factor is 5 for $l/k = 2$.) This property of wave-transient growth in shear flows was first pointed out by Orr (1907) and later by Rosen (1971), for the non-rotating case. Recently, Lu & Zeng (1981) and Farrell (1981) discussed its relevance to atmospheric flows. The mechanism for this amplification will be discussed in §7. Since the energy density eventually decays with time for $U't > l/k$, the initial growth is only transient in nature and does not lead to normal-mode instability.

A wave with the opposite initial tilt, i.e. $l/k < 0$, has a monotonically decreasing energy density for all time.

The aforementioned change in energy density occurs in time gradually and is distinct from the wave-energy change normally associated with (viscous) critical-level absorption.

(v) It can be shown that in the absence of viscosity there is *no* wave–mean-flow interaction for all $t > 0$, despite the presence of algebraic amplification and decay in the wave-energy density mentioned above. Specifically, the induced mean-flow acceleration, $\partial\bar{u}/\partial t$, which is equal to the momentum-flux divergence $-\partial\overline{uv}/\partial y$ due to the wave, is zero uniformly for all y and all $t > 0$, since $\overline{uv}$ for the plane wave is independent of y. It is, however, not appropriate to discuss the total energy budget for the present case of plane waves in an infinite domain, as the total wave energy is infinite. A more meaningful discussion of the energetics will be given in §7 for a wave packet, which is of compact support.

6. Large-time asymptotics of an inviscid solution; relation to normal-mode solution

The results of §4 in general, and §5 in particular, demonstrate that even in the absence of viscosity the initial-value solution does not develop a singularity in the domain if the initial condition is not singular. However, the normal-mode solution to the same inviscid equation, if it exists, *does* possess a critical-level singularity where $U(y)-c=0$. One can conclude from these facts that either (i) a normal-mode solution by itself is fictitious, i.e. it is not the steady-state limit of any initial-value problem, or (ii) it can only develop from an initial disturbance which is already singular at the critical level. We shall show in this section that (ii) is the case.

It should be remarked before proceeding that the apparent contradiction mentioned above associated with the steady-state solutions does not arise for the viscous case (no matter how small the viscosity is), as we have already shown that for that case the initial-value solution and the normal-mode solution (if the viscous continuation of Lin is used) both decay exponentially in time to the trivial solution. The problem of singularities in the steady-state solution arises only for purely inviscid flows. Therefore the issues to be addressed here are rather academic in nature; but the result will nevertheless serve to point out the origin of the normal-mode singularity and hence the unphysical nature of that representation.

6.1. *Large-time asymptotic limit*

The limit of the inviscid solution as $t \to \infty$ will be calculated for the case of constant shear, with the initial condition given by

$$\zeta(x,y,0) = \zeta_0(x,y) = \mathrm{e}^{\mathrm{i}k_0 x}\,\delta(y-y_0) \quad (k_0 > 0). \tag{6.1}$$

The solution for $t > 0$ is

$$\zeta(x,y,t) = \mathrm{e}^{\mathrm{i}k_0(x-U'ty)}\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}l\,\mathrm{e}^{\mathrm{i}l(y-y_0)}\exp\left\{\mathrm{i}\frac{\beta}{U'k_0}\left[\tan^{-1}\left(U't-\frac{l}{k_0}\right)+\tan^{-1}\frac{l}{k_0}\right]\right\}. \tag{6.2}$$

The integrand in (6.2) is not integrable in the usual sense. This is entirely to be expected because the initial spectrum in l is not integrable owing to the imposition of a delta function as the initial condition. For passing limits inside an integral sign it is more convenient to deal with an integrable integrand. For this purpose the stream function ψ will be treated first. Using (6.2) and (3.4), one finds that

$$\psi(x,y,t) = -\mathrm{e}^{\mathrm{i}k_0(x-U'yt)}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}l}{k_0^2+(l-U't)^2} \times\exp\left\{\mathrm{i}l(y-y_0)+\mathrm{i}\frac{\beta}{U'k_0}\left[\tan^{-1}\left(U't-\frac{l}{k_0}\right)+\tan^{-1}\frac{l}{k_0}\right]\right\}, \tag{6.3}$$

which becomes, upon a change in variable from l to $l' = l-k_0U't$,

$$\psi(x,y,t) = -\mathrm{e}^{\mathrm{i}k_0(x-U'y_0t)}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}l'}{k_0^2+l'^2} \times\exp\left\{\mathrm{i}\frac{\beta}{U'k_0}\left[-\tan^{-1}\frac{l'}{k_0}+\tan^{-1}\left(U't+\frac{l'}{k_0}\right)\right]\right\}. \tag{6.4}$$

We can now take the $t \to \infty$ limit inside the integral and find that the limit of

$$\frac{\psi(x,y,t)}{e^{ik_0(x-U'y_0t)}} \quad \text{as} \quad t \to \infty$$

is given by the integral

$$\phi(y) \equiv -\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{il'(y-y_0) - i\frac{\beta}{U'k_0}\left[\tan^{-1}\frac{l'}{k_0} - \frac{\pi}{2}\right]\right\}\frac{dl'}{k_0^2+l'^2}. \tag{6.5}$$

This integration can be performed exactly to yield a Whittaker function (see Gradshtyen & Ryzhik (1965, p. 405). Thus

$$-e^{-i\pi\gamma}\phi(y) = \begin{cases} \dfrac{1}{2k_0\Gamma(1+\gamma)} W_{\gamma,-\frac{1}{2}}(2k_0(y-y_0)), & (y-y_0>0), \\ \dfrac{1}{2k_0\Gamma(1-\gamma)} W_{-\gamma,-\frac{1}{2}}(2k_0(y_0-y)), & (y-y_0<0), \end{cases} \tag{6.6}$$

where $\gamma \equiv \beta/2U'k_0$ is assumed to be a positive non-integer. The Whittaker functions in (6.6) decay exponentially for large arguments, i.e.

$$W_{\pm\gamma,-\frac{1}{2}}(2|k_0(y-y_0)|) \sim e^{-|k_0(y-y_0)|}(2|k_0(y-y_0)|)^{\pm\gamma}, \quad (2|k_0(y-y_0)| \gg 1).$$

Therefore the solution satisfies the boundedness condition (2.3) at infinities. The function $\phi(y)$ is well-behaved everywhere except at $y = y_0$, which is the location of the singularity in the initial vorticity distribution. The singularity at y_0 is of the form $(y-y_0)\ln(y-y_0)$. Therefore the derivatives of $\phi(y)$ are unbounded at y_0, although $\phi(y)$ itself is finite and continuous everywhere.

To summarize, we have found that the initial-value problem with a delta-function distribution in the initial vorticity field approaches a steady state for large time of the form of a normal-mode solution:

$$\psi(x,y,t) \to e^{ik_0(x-ct)}\phi(y), \tag{6.7}$$

where c is here found to be given by the mean-flow speed at the location of the initial singularity, i.e. $c = U(y_0)$. The steady-state vorticity can also be found from

$$\zeta(x,y,t) \to e^{ik_0(x-ct)}\left[\frac{d^2}{dy^2}\phi - k_0^2\phi\right]. \tag{6.8}$$

Knowing the equation that the Whittaker functions in (6.6) satisfy, we find that

$$e^{-i\pi\gamma}\left[\frac{d^2}{dy^2}\phi - k_0^2\phi\right] = -\frac{2k_0\gamma}{y-y_0}\phi(y)\,e^{-i\pi\gamma} = \begin{cases} \dfrac{\gamma}{\Gamma(1+\gamma)(y-y_0)} W_{\gamma,-\frac{1}{2}}(2k_0(y-y_0)) & (y>y_0), \\ \dfrac{\gamma}{\Gamma(1-\gamma)(y-y_0)} W_{-\gamma,-\frac{1}{2}}(2k_0(y_0-y)) & (y<y_0), \end{cases} \tag{6.9}$$

and so the vorticity has a $1/(y-y_0)$ type of singularity at y_0.

6.2. *Normal-mode solution*

By making the *a priori* assumption that $\psi(x,y,t)$ is of the form

$$\psi(x,y,t) = e^{ik_0(x-ct)}\phi(y), \tag{6.10}$$

one finds that $\phi(y)$ satisfies the following normal-mode equation:

$$(U(y)-c)\left[\frac{d^2}{dy^2}\phi+\left(\frac{\beta}{U(y)-c}-k_0^2\right)\phi\right]=0. \tag{6.11}$$

Equation (6.11) implies that the Rayleigh–Kuo equation

$$\frac{d^2}{dy^2}\phi+\left(\frac{\beta}{U(y)-c}-k_0^2\right)\phi=0 \tag{6.12}$$

holds for all $y \neq y_c \equiv c/U'$, and that the following matching conditions across the critical level y_c should be used:

$$\phi(y_c+)=\phi(y_c-), \tag{6.13}$$

$$(U-c)\frac{d}{dy}\phi\bigg|_{y_c+}=(U-c)\frac{d}{dy}\phi\bigg|_{y_c-}. \tag{6.14}$$

The second condition, (6.14), is obtained by integrating (6.11) across $y = y_c$.

Solving (6.12) subject to the boundedness condition at infinities, we find

$$\phi(y)=\frac{A}{\Gamma(1+\gamma)}W_{\gamma,-\frac{1}{2}}(2k_0(y-y_c)),\quad (y>y_c), \tag{6.15}$$

$$\phi(y)=\frac{B}{\Gamma(1-\gamma)}W_{-\gamma,-\frac{1}{2}}(2k_0(y_c-y)),\quad (y<y_c). \tag{6.16}$$

Continuity of $\phi(y)$ at $y = y_c$ implies that

$$A=B, \tag{6.17}$$

since

$$W_{\gamma,-\frac{1}{2}}(z)\to\frac{1}{\Gamma(1+\gamma)}+O(|z\ln z|)\quad\text{as}\quad z\to 0.$$

The condition (6.14) gives only the identity that zero equals zero, and so yields no additional constraint on the solution. Therefore the final solution is

$$\phi(y)=\begin{cases}\dfrac{A}{\Gamma(1+\gamma)}W_{\gamma,-\frac{1}{2}}(2k_0(y-y_c)), & (y>y_c),\\[2ex] \dfrac{A}{\Gamma(1-\gamma)}W_{-\gamma,-\frac{1}{2}}(2k_0(y_c-y)), & (y<y_c).\end{cases} \tag{6.18}$$

The 'eigenvalue $c \equiv U'y_c$ is not uniquely determined: it can take any real value between $-\infty$ and ∞. From its definition and the location of the singularity in (6.18), one gives the interpretation for c only as the mean-flow speed at the location of singularity.

Comparing the normal-mode solution (6.18) with the long-time limit of the initial-value solution (6.6) and (6.7), one sees that the normal-mode solution, which is singular (in its vorticity distribution), is reachable from an initial value if the initial vorticity distribution is of the rather unphysical form of a delta function.†

† It should be pointed out, however, that there is no unique one-to-one correspondence between the final steady-state limit and the initial value, and frequently more than one initial condition can yield the same steady-state limit. A trivial example is the present problem with the Whittaker function instead of the delta function as the initial condition. Nevertheless, our result shows that no initial condition without a singularity can give rise to a singularity in its steady state.

Furthermore, the location of the critical level in the normal-mode solution should be identified with the location of the initial singularity.

We conclude therefore that, *for any bounded disturbance distribution, no critical-level singularity commonly associated with normal solutions will develop in the present stable free case. Conversely, the inviscid normal-mode solution, obtained with the* a priori *assumption of* (6.10), *is not the steady-state limit of a bounded initial-value solution for the present stable problem.*†

7. Wave-packet solutions

We now turn to the physically more relevant case where the initial spectrum is not flat (as it would be if the initial condition were a delta function). Since the generalization to the N-packet case is quite straightforward (see appendix C), we shall treat here only a representative case where the initial spectrum $\hat{\zeta}_0(k, l)$ is peaked about a central wavenumber

$$\boldsymbol{k}_0 \equiv (k_0, l_0).$$

But, unlike the plane-wave spectrum studied in §5, there is now some spectral spread $\Delta \boldsymbol{k}_0$ about the central wavenumber. Readers are referred to appendix C for details, where it is found that the general solution (3.6) reduces to

$$\zeta(x,y,t) = \zeta_0(\xi + \Omega_k^{(0)}, \eta + \Omega_l^{(0)}) \exp\{\mathrm{i}[\Omega^{(0)} - \Omega_k^{(0)} k_0 - \Omega_l^{(0)} l_0]\} \exp(-D^{(0)}) \quad (7.1)$$

up to the order $\Delta k_0/k_0$ and $\Delta l/l_0$. Here $\zeta_0(x, y)$ is the part of $\zeta(x, y, 0)$ that has a spectral peak at $\boldsymbol{k}_0$. It is understood that the contributions from other spectral peaks, if present, are to be added to (7.1). We have also defined

$$\Omega^{(0)} \equiv \Omega(k_0, l_0, t),$$

$$\Omega_k^{(0)} \equiv \frac{\partial}{\partial k}\Omega(k_0, l_0, t), \quad \Omega_l^{(0)} \equiv \frac{\partial}{\partial l}\Omega(k_0, l_0, t),$$

$$D^{(0)} \equiv D(k_0, l_0, t).$$

It is seen from (7.1) that the initial shape moves according to

$$\xi(t) = \xi(0) - \Omega_k^{(0)}, \quad \eta(t) = \eta(0) - \Omega_l^{(0)}. \quad (7.2)$$

Equation (7.2) describes the trajectory of the centre, say, of the disturbance whose initial position is $(\xi(0), \eta(0))$. Recalling that Ω can be written as the time-integral of

† As mentioned in §1, the normal-mode solutions, when superposed, can still yield a valid mathematical, though inconvenient, representation of any *bounded* solution from the initial-value problem. In the present simple example, the normal-mode solution (6.18) when superposed (i.e. integrated with respect to c from $-\infty$ to ∞) can give a valid representation of a solution evolved from any bounded initial condition. (This is basically a Laplace (or Fourier) transform method; see Case (1960) and Pedlosky (1964).) From another perspective, our delta-function initial condition can be viewed as the Green function of a more physical distribution, with the final solution to be obtained with an additional integration with respect to y_0, the location of the delta-function singularity. The same result can be obtained from either perspective. In the same way as one understands that the singularity in the integrand does not necessarily lead to a singularity in the integral, the singularity in the normal-mode solution in the present case should also be viewed as nothing more than a mathematical representation of an intermediate step in the solution process.

a frequency ω, one can express the time-dependent terms in (7.2) as

$$\left.\begin{aligned} \Omega_k(k,l,t) &= -\int_0^t c_{g1}(k,l,t')\,dt', \\ \Omega_k(k,l,t) &= -\int_0^t c_{g2}(k,l,t')\,dt', \end{aligned}\right\} \tag{7.3}$$

where $c_{g1} \equiv -\partial\omega/\partial k$ is the group velocity in the ξ-direction, and $c_{g2} \equiv -\partial\omega/\partial l$ is the group velocity in the η-direction. Since $\omega(k,l,t) = \beta k/[k^2+(l-kT(t))^2]$, we deduce

$$\left.\begin{aligned} c_{g1}(k,l,t) &= \frac{\beta[k^2(1+T(t)^2)-l^2]}{[k^2+(l-kT(t))^2]^2}, \\ c_{g2}(k,l,t) &= \frac{2\beta k(l-kT(t))}{[k^2+(l-kT(t))^2]^2}. \end{aligned}\right\} \tag{7.4}$$

At large $T(t)$, both group velocities vanish; the packets must stagnate (in the convected coordinates), as can be verified.

As an explicit example, we consider the constant-shear case, for which $T(t) = U't$, and so from (3.11) we find that the trajectories (7.2) become

$$\xi(t) = \xi(0) + \frac{\beta}{U'k_0^2}\left[\tan^{-1}\left(U't - \frac{l_0}{k_0}\right) + \tan^{-1}\frac{l_0}{k_0} - \frac{l_0/k_0}{1+(U't-l_0/k_0)^2} + \frac{l_0/k_0}{1+(l_0/k_0)^2}\right], \tag{7.5}$$

$$\eta(t) = \eta(0) + \frac{\beta}{U'k_0^2}\left[\frac{1}{1+(U't-l_0/k_0)^2} - \frac{1}{1+(l_0/k_0)^2}\right]. \tag{7.6}$$

As $U't \to \infty$, both $\xi(t)$ and $\eta(t)$ have finite limits, implying the existence of stagnation levels in both ξ- and η-directions beyond which the centre of the wave packet does not extend. This is consistent with the group-velocity results mentioned earlier.

It is clear from (7.6) (and also from (7.4)) that the packet with positive l_0/k_0 has a very different trajectory than the packet with negative l_0/k_0. For $l_0/k_0 > 0$, the initial shape moves first northward until it reaches a turning point at η_T given by

$$\eta_T = \eta(0) + \frac{\beta}{U'}\left[\frac{1}{k_0^2} - \frac{1}{k_0^2+l_0^2}\right]. \tag{7.7}$$

This occurs at time

$$U't = \frac{l_0}{k_0}, \tag{7.8}$$

when the group velocity in the meridional direction vanishes. After this time the group velocity becomes negative. The packet then moves southward, eventually stagnating at

$$\eta_c \equiv \eta(\infty) = \eta(0) - \frac{\beta/U'}{k_0^2+l_0^2}. \tag{7.9}$$

For the case where l_0 and k_0 are of different signs, the packet trajectory is monotone; the initial shape moves southward without changing direction towards the same stagnation level (7.9) as the previous case. The trajectories are depicted in figure 2. Similar trajectories have previously been obtained by Yamagata (1976*b*) using the ray-tracing method. Note the following results:

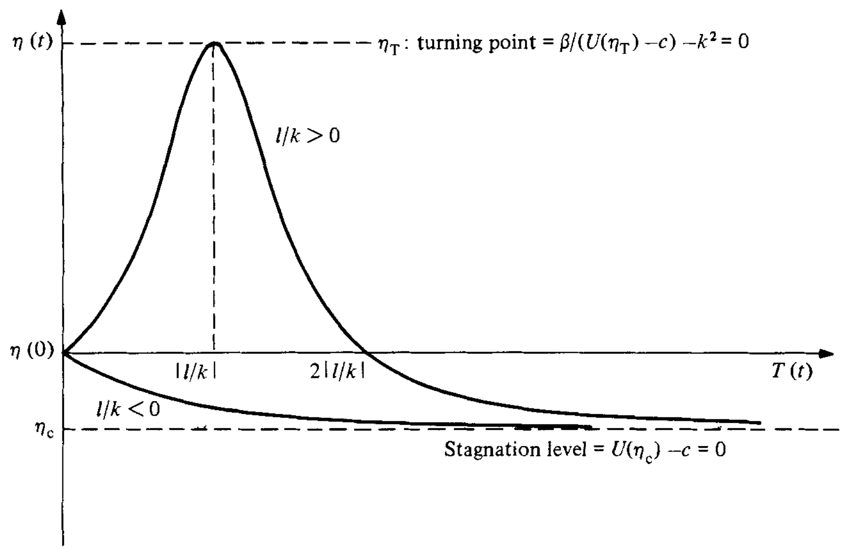

FIGURE 2. Northward packet trajectories $\eta(t)$ or $y(t)$ *vs.* normalized time $T(t) = U't$ for a packet with central wavenumbers k and l of the same sign, and also for a packet whose central wavenumbers are of opposite signs.

(i) *Kinematics.* The stagnation level occurs in this problem where the mean flow equals the *unsheared* Rossby-wave phase speed, i.e. where

$$U(\eta_c) = c, \tag{7.10}$$

with

$$c = U(\eta(0)) - \frac{\beta}{k_0^2 + l_0^2} \tag{7.11}$$

being the barotropic Rossby wave speed in an atmosphere with a constant mean flow equal to the flow at the initial point $\eta(0)$ in this problem. Of course, the actual phase speed of the wave varies with time; only the kinematic behaviour of the packet resembles that of a Rossby wave with a wave speed c.

(ii) *Wave-energy density.* The averaged energy density of the packet is found to be

$$\overline{K}(t) = \tfrac{1}{2}\{\overline{|\psi_x|^2 + |\psi_y|^2}\} = \frac{1}{4}\frac{|\zeta_0(\xi + \Omega_k^{(0)}, \eta + \Omega_l^{(0)})|^2 \, e^{-2D^{(0)}}}{k_0^2 + (l_0 - k_0 U't)^2}. \tag{7.12}$$

An observer moving with the packet

$$\xi + \Omega_k^{(0)} = \xi(0), \quad \eta + \Omega_l^{(0)} = \eta(0)$$

sees an energy density given by

$$\overline{K}(t) = \frac{\tfrac{1}{2}|a|^2 \, e^{-2D^{(0)}}}{k_0^2 + (l_0 - k_0 U't)^2}, \tag{7.13}$$

with $a \equiv \zeta_0(\xi(0), \eta(0))$ being the initial amplitude of the vorticity distribution. Equation (7.13) turns out to be the same as (5.8) for the plane-wave case, and so its behaviour is the same as depicted in figure 1. Thus if $l_0/k_0 > 0$, the northward-moving packet in the inviscid case gains energy density algebraically, reaching a maximum factor of $1 + (l_0/k_0)^2$ over the initial energy density. This maximum occurs at the turning point η_T given by

$$\frac{\beta}{U(\eta_T)-c}-k_0^2=0. \tag{7.14}$$

For $U't > l_0/k_0$, the packet energy density decreases algebraically as the packet moves southward. If $l_0/k_0 < 0$, the southward-moving packet has a monotonically decreasing energy density.

(iii) *Wave action and momentum flux.* The wave action $\overline{A} = k\,\overline{K}/\omega$ for a wave packet is found to be

$$\overline{A}=\frac{1}{4\beta}\mathrm{e}^{-2D^{(0)}}|\bar{\zeta}_0(\eta+\Omega_l^{(0)})|^2. \tag{7.15}$$

(The slow variations in the zonal direction are not considered in the following discussion in order to be consistent with other studies where the overbar is taken to be the zonal average over all x-dependence.) It is seen from (7.15) that, in the absence of damping, the action is conserved following the packet. In the presence of damping, it is the quantity $\overline{A}\,\mathrm{e}^{2D^{(0)}}$ that is conserved. That is, $\overline{A}$ from (7.15) satisfies

$$\left(\frac{\partial}{\partial t}+c_{g2}\frac{\partial}{\partial y}\right)(\overline{A}\,\mathrm{e}^{2D^{(0)}(t)})=0, \tag{7.16}$$

where the group velocity c_{g2} is defined in (7.4).

The zonally averaged meridional momentum flux is calculated from our solution to be

$$\overline{uv}=-\frac{\frac{1}{2}k_0(l_0-k_0U't)\,\mathrm{e}^{-2D^{(0)}}}{[k_0^2+(l_0-k_0U't)^2]^2}|\bar{\zeta}_0(\eta+\Omega_l^{(0)})|^2.$$

It becomes, if (7.4) and (7.15) are used,

$$\overline{uv}=-c_{g2}(t)\,\overline{A}(y,t). \tag{7.17}$$

The momentum-flux divergence is thus given by

$$\frac{\partial}{\partial y}\overline{uv}=\mathrm{e}^{-D^{(0)}(t)}\frac{\partial}{\partial t}(\overline{A}\,\mathrm{e}^{2D^{(0)}(t)}) \tag{7.18}$$

after (7.16) is used. The relation in (7.18) is similar to the one first derived by Acheson (1976) for gravity waves in the absence of dissipation assuming slow time variation.

(iv) *Induced mean flow.* Let $\bar{u}$ be the wave-induced mean flow at second order in wave amplitude. It is given by the following equation obtained by zonally averaging the x-momentum equation:

$$\frac{\partial}{\partial t}\bar{u}=-\frac{\partial}{\partial y}(\overline{uv}). \tag{7.19}$$

An extra term, $\nu\,\partial^2\bar{u}/\partial y^2$, which is at least two orders of magnitude (in $\Delta l_0/l_0$) smaller than the terms retained in (7.19), has been neglected in that equation. Using (7.18), we find that the induced acceleration is

$$\frac{\partial}{\partial t}\bar{u}=-\mathrm{e}^{-2D^{(0)}(t)}\frac{\partial}{\partial t}(\overline{A}\,\mathrm{e}^{2D^{(0)}(t)}). \tag{7.20}$$

Let us first discuss the situation in the absence of viscosity. Equation (7.20) can be integrated (see also Acheson 1976), yielding

$$\bar{u}(y,t)=-\overline{A}(y,t)\quad(t\geqslant 0), \tag{7.21}$$

assuming that there is no disturbance for $t < 0$. As the disturbance is introduced at $t = 0$, the mean flow is first decelerated by the amount $\overline{A}(y, 0)$ in the region of the initial disturbance. After the packet leaves that region, $\overline{A}(y, t)$ becomes zero and the mean flow is restored to the undisturbed value by the trailing part of the packet. Similar arguments applied to other parts of the fluid would show that at any location there would be first a deceleration as the leading part of the packet arrives and an acceleration on its departure, leaving *no* permanent alteration of the mean flow. The only exception occurs at the stagnation level, where there will be a permanent reduction of the mean flow by the amount

$$\bar{u} = -\overline{A}(y_c, \infty)$$

as the packet stagnates (and hence cannot depart from that level). This momentum is of the same amount as originally introduced by the initial disturbance but has been redistributed by the wave packet to its final destination.

It will be shown next that, even though the momentum of the mean flow is conserved for $t > 0$, the energy density of the mean flow is not, and this is the cause of the transient change in the wave-energy density.

(v) *Net wave-energy density.* The total (kinetic) energy density (per unit mass) of the system is

$$\tfrac{1}{2}(U+\bar{u})^2 + \overline{K} \simeq \tfrac{1}{2}U^2 + \bar{u}U + \overline{K},$$

where $U = U'(y - y_0)$ is the *pre-existing* flow, $\bar{u}$ the wave-induced mean flow calculated in (7.21) and $\overline{K}$ is the wave kinetic-energy density (7.12). The *net* wave-energy density E, defined by the difference between the total energy density and the energy density of the pre-existing (undisturbed) flow, is thus given by (see e.g. Acheson 1976)

$$E = \bar{u}U + K. \tag{7.22}$$

Since $\overline{K} = \overline{A}\omega/k_0$ and $\bar{u} = -\overline{A}$, we find

$$\begin{aligned} E &= \overline{A}\left(\frac{\omega}{k_0} - U\right) \\ &= \overline{A}\left[\frac{\beta}{k_0^2 + (l_0 - k_0 U' t)^2} - U'\eta\right]. \end{aligned} \tag{7.23}$$

Equation (7.23) implies that the net energy density E is constant to an observer moving with the wave packet. This is because the action is conserved and the terms in the square brackets are equal to

$$\frac{\beta}{k_0^2 + l_0^2} - U'\eta(0),$$

when the packet trajectory (7.6) is followed. Thus

$$E(t) = -c\overline{A}|_{t=0} = E(0) \tag{7.24}$$

following the packet.

The transient amplification and subsequent decay of the wave-energy density $\overline{K}(t)$ mentioned previously can now be explained. Since in (7.22) both E and $\bar{u}$ are conserved following the packet, the change in $\overline{K}(t)$ can only be attributed to the change in $U(\eta(t))$ as the packet moves north or south. As the packet moves north from its original position, the induced deceleration $\bar{u}$ occurs in a region with a higher value of U, causing a greater reduction in mean-flow energy density $\bar{u}U$. To conserve net energy density E, the wave-energy density $\overline{K}$ must increase. In the packet's subsequent southward journey after reflection from the turning point, the induced deceleration occurs in a region with a lower value of U, causing a smaller reduction in mean-flow energy density (than the initial reduction). $\bar{K}$ then must decrease. As the stagnation level is approached, $\bar{u}U$ approaches $\bar{u}c$, which is equal to the total E (see (7.24)). Thus $\bar{K}$ must vanish.

The type of wave–mean-flow interaction discussed above is distinct from that commonly associated with 'critical-level absorption'. The latter is caused by wave dissipation in the presence of viscosity, to be discussed in (vii).

(vi) *Instabilities.* Since the temporal amplification mentioned above is associated with a moving wave group, it is different from what is called 'absolute' or 'normal-mode' instability. In an absolute instability, a disturbance field grows in time at all space points, while, in the present case, the growth does not occur everywhere, but is observed when moving with the packet.

Furthermore, the algebraic growth of the wave-energy density is not sustained. After reflection from the turning point, the wave-energy density decays monotonically in time as the packet moves southward to regions with lower values of the mean flow. The transient time growth that is observed initially does not lead to normal-mode instability for this case.

The situation would be very different if the southward journey of the wave packet could be stopped. If the 'inflection point' is encountered before the stagnation level, not only is the decay in wave-energy density arrested, but, by turning the packet northward, the time amplification mentioned in (ii) can again be realized. How this procedure can lead to normal-mode instability is not at all clear at present, but at least we know that, without the 'inflection point', no sustained amplification can occur. This argument is consistent with Rayleigh's (1880) derived result that the presence of an 'inflection point' is *necessary* for normal-mode instability. That Rayleigh's criterion is not *sufficient* for normal-mode instability can be demonstrated from the above perspective for the case where the 'inflection point' occurs to the south of the stagnation level, and is therefore not effective in preventing the energy-density decay from taking its full course. It is perhaps not a coincidence that a more stringent necessary condition for normal-mode instability, due to Fjørtoft (1950), does require the 'inflection point' to occur in the region where $U(y)-c$ is positive for the present monotonic profile (see Lindzen & Tung 1978). Recently Tung (1981) has shown that this requirement is also *sufficient* for normal-mode instability (for the present case of monotonic profiles in infinite domains).

(vii) *Viscous absorption.* In addition to the algebraic time variations in the wave's energy density, mentioned in (ii) and (v), caused by transient wave–mean-flow interaction, there is a second mechanism, the action of viscosity, causing the decay of the wave's energy density, according to (7.12). There is, however, no 'singular absorption' at a certain location *in space* commonly associated with a critical level. Instead, the dissipation by viscosity acts continuously *in time*. The exponential factor in (7.1), $e^{-D^{(0)}(t)}$, diminishes the wave amplitude according to the *age* of the wave since generation.

However, since the packet eventually stagnates at the stagnation level, most of the *viscous dissipation* does occur at that location, provided that viscosity is so small that the dissipation of the wave *en route* is insignificant, and that one is examining the wave field at a time late enough after generation that the waves have already spent a sufficiently long time in the vicinity of the stagnation level for the viscosity to act on them. This then constitutes an interpretation, from an initial-value approach, of the 'critical-level-absorption' result of Dickinson (1969). It should be pointed out, however, that, in order for this argument to work, viscosity has to be so small that viscous dissipation *en route* can be ignored, but, for this case, the transient wave–mean-flow interaction discussed in (v) becomes the dominant mechanism for wave 'absorption'. Our earlier result in (v) shows that, by the time the wave packet approaches the vicinity of the stagnation level, most of its wave-energy density has been converted into mean-flow energy. Specifically, the total initial (net) wave energy E in (7.24) resides now mostly in the mean-flow part $\bar{u}U$ (see (7.22)) by the time the wave packet approaches the stagnation level. In other words, there is not much wave energy left for the viscosity to 'absorb' at the critical level. It can also be shown, using (7.20), that the additional 'absorption' of wave momentum by viscosity is also negligible at the stagnation level.

For most practical cases, the damping timescale is about a week, which is comparable to the length of time it takes the packet to travel to the vicinity of the stagnation level. Therefore it appears that wave dissipation *en route* is important and that the role of the critical level in 'absorbing' the wave energy at the wave's final destination is probably not important in practical situations. It can be argued then that the critical level in the stable–free case plays no special role, and can be replaced by, say, a wall, provided one can somehow simulate the kinematic propagational characteristics of the waveguide.

8. Conclusion

We have used a Rossby-wave problem as an example to demonstrate that the normal-mode solution with critical-level behaviour in some cases bears no resemblance to the initial-value problem, not even in the infinite-time limit. It is pointed out that it is misleading to use the scaling based on the normal-mode structure to determine, for example, where nonlinear terms would become dominant. It will be shown, in a separate paper, however, that the solution to the mathematical initial-value problem where forcing is continuously present would approach the normal-mode singular structure at infinite time in the absence of viscosity. It is suggested that, for the free-wave problem and also for the forced-wave problem where forcing is present for a finite period of time, it is more appropriate to treat the mathematical problems as time-dependent rather than to invoke the normal-mode separation (1.1).

The simple solutions obtained serve to demonstrate the two separate mechanisms through which wave–mean-flow interaction can take place in an initial-value problem: viscosity provides one mechanism for 'absorbing' the wave's energy. Although the 'absorption' actually occurs continuously in time after the wave's generation, as opposed to the concept of 'singular absorption' that occurs in a particular location as implied by the normal-mode approach, the two viewpoints can be reconciled in the asymptotic limit of small viscosity, as already discussed. The second mechanism involves the type of energy exchange that conserves wave action. It is consistent with the asymptotic result of Bretherton & Garrett (1969) on wave propagation in a slightly inhomogeneous medium. (It turns out that the inhomogeneity, i.e. shear, does not need to be small for the present problem.) Wave amplification occurs following the packet as the easterly momentum of the packet is deposited (temporarily) in the region where the pre-existing mean-flow velocity is higher; the wave-energy density decays when the same easterly momentum is deposited in a region of lower mean-flow velocity. It is argued that in initial-value problems this mechanism of energy transfer is more important than that of critical-level absorption in the presence of small viscosity. For finite values of viscosity used in practical situations, viscous dissipation in time may become more important. However, for this case, the interpretation of critical-level absorption is less clear in an initial-value free problem. In any case, the critical level's role in absorbing wave energy becomes secondary.

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Appendix A. Boundary conditions

The solution (3.6) with $\hat{\zeta}(k,l,\tau)$ given by (3.8) is valid in an infinite domain, satisfying the boundary conditions (2.3) with $L\rightarrow\infty$ and (2.4*b*). If the condition of periodicity in the zonal (x-) direction, (2.4*a*), is used instead, the integral in k in (3.4) should be replaced by a sum, i.e.

$$\int_{-\infty}^{\infty} \mathrm{d}k \rightarrow \sum_{n=-\infty}^{\infty} \Delta k, \quad \Delta k = \frac{1}{a\cos\theta_0}, \tag{A 1a}$$

and the continuous wavenumber k replaced by a set of discrete wavenumbers, i.e.

$$k \rightarrow k_n \equiv n\Delta k \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{A 1b}$$

By virtue of the integrability condition, the original solution (3.6) satisfies the meridional boundary condition

$$\zeta \rightarrow 0 \quad \text{as} \quad y \rightarrow \pm\infty,$$

which is (2.3) with $L\rightarrow\infty$. In a finite β-plane with finite L, the solution has to be modified slightly in order to satisfy the boundary conditions

$$\zeta = 0 \quad (y = \pm L). \tag{A 2}$$

The modification again involves replacing the continuous wavenumber l by its discrete counterpart

$$l \rightarrow l_n \equiv n\,\Delta l \quad (n = 1, 2, 3, \ldots), \tag{A 3a}$$

with

$$\Delta l = \frac{\pi}{2L} \tag{A 3b}$$

for this case. However, the index n takes only positive values in (A 3*a*). Therefore the solution $\hat{\zeta}(k,l,\tau)$ in (3.8) is now defined only for the positive discrete points l_n. The function at the negative values of l is defined through the formula

$$\hat{\zeta}(k,l,\tau) = -\mathrm{e}^{-2\mathrm{i}lL}\hat{\zeta}(k, -l, \tau). \tag{A 4}$$

With this continuation formula, we have in effect made the original solution (3.2) odd about the point y (or η) $= -L$. This is permissible since the domain in $y > -L$ is outside the physical domain. To show that with the modifications in (A 3*a*) and (A 4) the boundary condition (A 2) is satisfied, we note that the meridional Fourier integral now becomes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}l\, \mathrm{e}^{\mathrm{i}ly}\, \hat{\zeta}(k,l,\tau) = \frac{\mathrm{i}}{\pi}\int_{0}^{\infty} \mathrm{d}l \sin l(y+L)\, \mathrm{e}^{-\mathrm{i}lL}\, \hat{\zeta}(k,l,\tau)$$

$$= \frac{\mathrm{i}}{\pi}\sum_{n=0}^{\infty} \Delta l \sin l_n(y+L)\, \mathrm{e}^{-\mathrm{i}l_n L}\, \hat{\zeta}(k,l_n,\tau) \qquad \text{(A 5)}$$

which vanishes at $y = \pm L$, as required. Note that to be consistent with the above formulation the initial spectrum $\hat{\zeta}_0(k,l)$ is given by the Fourier transform of the initial condition only for positive values of $l = l_n$:

$$\hat{\zeta}_0(k,l_n) = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}kx-\mathrm{i}l_n y}\, \zeta_0(x,y). \qquad \text{(A 6)}$$

The function at negative values $l = -l_n$ should be obtained through the continuation (A 4). The same procedure should also be followed for $\hat{\zeta}(k,l_n,\tau)$.

Appendix B. Nonlinearity

B.1. *Uniform boundedness of the leading-order solution for $\epsilon \rightarrow 0$*

We first need to show that the solution obtained in §3 by neglecting nonlinearity is uniformly bounded, i.e. it is non-singular everywhere. It suffices to consider the boundedness issue for the inviscid problem, as the viscous term would always make the solution more bounded for large enough time because D is positive.†

Using Parseval's formula, it is easy to show that the solution

$$\zeta(x,y,t) = \frac{1}{(2\pi)^2}\iint \hat{\zeta}_0(k,l)\, \mathrm{e}^{\mathrm{i}k\xi+\mathrm{i}l\eta+\mathrm{i}\Omega}\, \mathrm{d}k\, \mathrm{d}l \qquad \text{(B 1)}$$

is bounded in the mean, in the sense that the integral of $|\zeta|^2$ over the domain is bounded. Since

$$\iint |\zeta|^2\, \mathrm{d}x\, \mathrm{d}y = \frac{1}{(2\pi)^2}\iint |\hat{\zeta}(k,l,t)|^2\, \mathrm{d}k\, \mathrm{d}l$$

$$= \frac{1}{(2\pi)^2}\iint |\hat{\zeta}_0(k,l)|^2\, \mathrm{d}k\, \mathrm{d}l$$

$$= \iint |\zeta(x,y,0)|^2\, \mathrm{d}x\, \mathrm{d}y, \qquad \text{(B 2)}$$

one can conclude that the solution for all time is bounded in the mean if it is so initially. This, however, does not imply uniform boundedness, for which we need to show that $|\zeta(x,y,t)|$ itself is bounded *everywhere*. Applying Schwarz's inequality for infinity integrals (or sums when appropriate), one obtains from (B 1)

$$|\zeta(x,y,t)| \leqslant \frac{1}{(2\pi)^2}\iint |\hat{\zeta}_0(k,l)\, \mathrm{e}^{\mathrm{i}k\xi+\mathrm{i}l\eta+\mathrm{i}\Omega}|\, \mathrm{d}k\, \mathrm{d}l$$

$$= \frac{1}{(2\pi)^2}\iint |\hat{\zeta}_0(k,l)|\, \mathrm{d}k\, \mathrm{d}l. \qquad \text{(B 3)}$$

† Marcus & Press (1977) previously treated the non-rotating case of the present problem and showed that the tangential velocity is uniformly bounded from above by a constant times $(\nu t)^{-\frac{1}{2}}$. This bound is not tight enough for our purpose as it approaches infinity for finite times in the limit $\nu \rightarrow 0$.

We conclude therefore that if the *initial* Fourier spectrum (which is the same in convected or fixed coordinates) is *absolutely integrable*, then the solution *for all later time* is uniformly bounded and hence no singularity can develop. Furthermore, since absolute integrability is a prerequisite for the existence of a Fourier inversion, and since all physical initial distributions will have such a spectrum, it appears that our initial-value solution will not develop a singularity in the flow field, in particular, not the critical-level singularity associated with the normal-mode solution.

The proof can also be applied to the stream function with only slight modifications. The solution for ψ is

$$\psi(x,y,t) = -\frac{1}{(2\pi)^2}\iint \frac{\hat{\zeta}_0(k,l)}{k^2+(l-kT)^2}\, \mathrm{e}^{\mathrm{i}k\xi+\mathrm{i}l\eta+\mathrm{i}\Omega}\,\mathrm{d}k\,\mathrm{d}l, \tag{B 4}$$

which implies the following inequality:

$$|\psi(x,y,t)| \leqslant \frac{1}{(2\pi)^2}\iint \frac{\hat{\zeta}_0(k,l)}{k^2+(l-kT)^2}\,\mathrm{d}k\,\mathrm{d}l. \tag{B 5}$$

The maximum of (B 5) occurs when $T = l/k$ (which, incidentally, is the time when the wave with positive l/k reaches its turning point – see §7). Thus

$$|\psi(x,y,t)| \leqslant \frac{1}{(2\pi)^2}\iint \frac{\hat{\zeta}_0(k,l)}{k^2}\,\mathrm{d}k\,\mathrm{d}l. \tag{B 6}$$

The uniform boundedness of ψ then follows if $\hat{\zeta}_0(k,l)/k^2$ is absolutely integrable. Since the initial spectrum for vorticity is absolutely integrable, $\hat{\zeta}_0(k,l)/k^2$ should also be. The only problem may arise at $k=0$ if $\hat{\zeta}_0(k,l)$ does not vanish as k approaches zero. We know, however, $\hat{\zeta}_0(0,l)$ *is* zero by the definition of *wave disturbance* used in this study. The zonally uniform component of the flow has been assumed to be $U(y)$, which is not modified at this order in the asymptotic expansion. The question that remains is whether $\hat{\zeta}_0(k,l)$ approaches zero at least as fast as k^2 for small k. We will assume that this is the case. In fact, in physical problems, disturbances with very long zonal wavelengths should be constrained by the periodicity of the zonal circle, which quantizes k into discrete values, with $\hat{\zeta}_0(k,l) \equiv 0$ in the neighbourhood of $k=0$.

It has often been pointed out (see Farrell 1982) for a plane wave, or a wave packet with an initial central normal wavenumber much larger than its central tangential wavenumber, that the transient amplification implied by the factor in the denominator of the above expression will lead to an amplification factor of $1+(l/k)^2$, which can be so large as to invalidate the linear assumption. Our discussion here suggests that this does not happen because

$$\hat{\psi}_0(k,l)\left(1+\left(\frac{l}{k}\right)^2\right) = \frac{-\hat{\zeta}_0(k,l)}{k^2},$$

and $\hat{\zeta}_0(k,l)/k^2$ should be bounded.

B.2. *Nonlinear correction*

To find the nonlinear correction to the linear solution (B 1), we write the nonlinear solution to (2.1) in the form

$$\zeta(x,y,t) = \zeta^{(0)}(x,y,t)+\epsilon\zeta^{(1)}(x,y,t), \tag{B 7}$$

where $\zeta^{(0)}$ is the linear solution satisfying (3.1) and $\epsilon\zeta^{(1)}(x,y,t)$ the required nonlinear correction. Substituting (B 7) into the nonlinear equation (2.1), one gets, to order ϵ, the following inhomogeneous equation for $\zeta^{(1)}$:

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\zeta^{(1)}+\beta\frac{\partial}{\partial x}\psi^{(1)}-\nu\nabla^2\zeta^{(1)} = -\mathrm{J}[\psi^{(0)},\zeta^{(0)}], \tag{B 8}$$

subject to the initial condition

$$\zeta^{(1)}(x,y,0)=0. \tag{B 9}$$

Using the linear viscous solution, we know already the 'forcing term' on the right-hand side of (B 8):

$$\begin{aligned}-\mathrm{J}[\psi^{(0)},\zeta^{(0)}] &= -\psi_\xi^{(0)}\zeta_\eta^{(0)}+\psi_\eta^{(0)}\zeta_\xi^{(0)}\\ &= \frac{1}{(2\pi)^4}\int \mathrm{d}k'\int \mathrm{d}k''\int \mathrm{d}l'\int \mathrm{d}l''\,\frac{\hat{\zeta}_0(k',l')\,\hat{\zeta}_0(k'',l'')}{1+(T(\tau)-l'/k')^2}\,\frac{k''l'-k'l''}{k'^2}\exp\{\mathrm{i}(k'+k'')\xi\\ &\quad +\mathrm{i}(l'+l'')\eta+\mathrm{i}\Omega(k',l',\tau)+\mathrm{i}\Omega(k'',l'',\tau)-D(k'l',\tau)-D(k'',l'',\tau)\}. \end{aligned} \tag{B 10}$$

Note that the integrand decays in time exponentially for the viscous case and algebraically for the inviscid case.

Incidentally, because of the presence of the factor $k''l'-k'l''$ in the integrand, one sees that the nonlinear 'forcing term' vanishes when

$$k'+k''=0,\quad l'+l''=0 \tag{B 11}$$

or

$$k'-k''=0,\quad l'-l''=0. \tag{B 12}$$

If the initial spectrum consists of only waves that satisfy (B 11) or (B 12) then the nonlinear term vanishes identically, and the linear solution also satisfies the nonlinear equation (2.1) exactly. For example, the solution satisfying the plane-wave initial condition

$$\zeta(x,\,y,\,0)=a\cos\,(kx+ly) \tag{B 13}$$

is given by, from (3.6) and (3.8),

$$\zeta(x,y,t)=a\cos\left(k\left(x-\int_0^t U(y,t)\,\mathrm{d}t\right)+ly+\Omega(k,l,t)\right)\exp\{-D(k,l,t)\}. \tag{B 14}$$

It can be shown by direct substitution that (B 14) is also the exact solution to the nonlinear viscous equation (2.1). Equation (B 14) appears to be the only known exact solution to the nonlinear viscous barotropic vorticity equation in the presence of a sheared flow.

Returning now to the more general case where the Jacobian term is not zero, we proceed to calculate the nonlinear correction $\zeta^{(1)}$ forced by the Jacobian of the leading-order solution. Using the convolution theorem, we find that the Fourier transform of (B 10) is

$$\begin{aligned}&-\int \mathrm{d}\xi\int \mathrm{d}\eta\,\mathrm{e}^{-\mathrm{i}k\xi-\mathrm{i}l\eta}\,\mathrm{J}(\psi^{(0)},\zeta^{(0)})\\ &= \frac{1}{(2\pi)^2}\int \mathrm{d}k'\int \mathrm{d}l'\,\hat{\zeta}_0(k',l')\,\hat{\zeta}_0(k-k',l-l')\frac{kl'-k'l}{k'^2[1+(T(\tau)-l'/k')^2]}\\ &\quad\times\exp\{\mathrm{i}\Omega(k',l',\tau)-D(k',l',\tau)+\mathrm{i}\Omega(k-k',l-l',\tau)-D(k-k',l-l',\tau)\}\\ &\equiv B(k,l,\tau). \end{aligned} \tag{B 15}$$

Writing

$$\zeta^{(1)}(x,y,t)=\frac{1}{(2\pi)^2}\int \mathrm{d}k\int \mathrm{d}l\,\mathrm{e}^{\mathrm{i}k\xi+\mathrm{i}l\eta}\,\hat{\zeta}^{(1)}(k,l,\tau) \tag{B 16}$$

and taking the Fourier transform of (B 8), one arrives at the following ordinary differential equation for $\hat{\zeta}^{(1)}$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\zeta}^{(1)}+\frac{\mathrm{i}\beta}{k}\left[1+\left(T(\tau)-\frac{l}{k}\right)^2\right]^{-1}\hat{\zeta}^{(1)}+\nu k^2\left[1+\left(T(\tau)-\frac{l}{k}\right)^2\right]\hat{\zeta}^{(1)}=B(k,l,\tau). \quad (\mathrm{B}\ 17)$$

The solution to (B 17) satisfying the initial condition (B 9) is

$$\hat{\zeta}^{(1)}(k,l,\tau)=\mathrm{e}^{\mathrm{i}\Omega(k,l,\tau)-D(k,l,\tau)}\int_0^\tau \mathrm{d}\tau'\,B(k,l,\tau')\,\mathrm{e}^{-\mathrm{i}\Omega(k,l,\tau')+D(k,l,\tau')}. \quad (\mathrm{B}\ 18)$$

The objective here is to determine if $\zeta^{(1)}(k, l, \tau)$ remains bounded and of order unity or less for all time.

Let us first treat the case of $\nu = 0$. One finds from (B 18) that

$$|\hat{\zeta}^{(1)}(k,l,\tau)| \leqslant \int_0^\tau \mathrm{d}\tau'\,|B(k,l,\tau')| = \frac{1}{(2\pi)^2}\int_0^\tau \mathrm{d}\tau'\int \mathrm{d}k'\int \mathrm{d}l'\,|\hat{\zeta}_0(k',l')\,\hat{\zeta}_0(k-k',l-l')| \times\frac{|kl'-k'l|}{k'^2[1+(T(\tau')-l'/k')^2]}. \quad (\mathrm{B}\ 19)$$

Since the integrand is integrable with respect to k' and l', one can interchange the order of integration and write (B 19) as

$$|\hat{\zeta}^{(1)}(k,l,\tau)| \leqslant \frac{1}{(2\pi)^2}\int \mathrm{d}k'\int \mathrm{d}l'\int_0^\tau \frac{\mathrm{d}\tau'\,|kl'-k'l|}{k'^2[1+(T(\tau')-l'/k')^2]}|\hat{\zeta}_0(k',l')\,\hat{\zeta}_0(k-k',l-l')|. \quad (\mathrm{B}\ 20)$$

The τ-integral is bounded for all τ for fixed k, l, k' and l' for monotonic $T(\tau)$. As an illustration, it is easily seen that for the constant-shear case

$$T(\tau') = U'\tau'$$

the integral can be performed exactly, yielding

$$\int_0^\tau \frac{\mathrm{d}\tau'}{1+(U'\tau'-l'/k')^2}=\frac{1}{U'}\left[\tan^{-1}\left(U'\tau-\frac{l'}{k'}\right)+\tan^{-1}\frac{l'}{k'}\right],$$

which is seen to be bounded for all τ. In addition, it is bounded for all k' and l'. Therefore the integral on the right-hand side of (B 20) exists and is bounded for all time. This implies the uniform boundedness of the *spectrum* $\hat{\zeta}^{(1)}(k,l,\tau)$.

To show that $\zeta^{(1)}(x,y,t)$ is uniformly bounded we need to demonstrate the absolute integrability of $\hat{\zeta}^{(1)}(k,l,\tau)$ with respect to k and l. By going inside in the integral sign in (B 20), which is permissible, one can see that $\hat{\zeta}^{(1)}(k,l,\tau)$ is absolutely integrable with respect to k and l provided that the quantity

$$\hat{\zeta}_0(k-k',l-l')\,[(k-k')\,l'-(l-l')\,k']$$

is absolutely integrable with respect to k and l. Thus we have shown that the nonlinear correction is uniformly bounded and hence remains of order ϵ for all time if the spectrum of the initial disturbance is such that $\hat{\zeta}_0(k,l)$ decays to zero for large k and l faster than k^{-2} and l^{-2}. This condition is slightly more stringent than the condition for absolute integrability, but is nevertheless satisfied by almost all physically reasonable initial conditions.

Some of the arguments used above do not hold when viscosity is retained. However, it can be shown that, since

$$D(k,l,\tau) \geqslant D(k,l,\tau') \quad (\tau \geqslant \tau')$$

for monotonic $T(\tau)$, we have from (B 15)

$$\begin{aligned}|\zeta^{(1)}(k,l,\tau)| &\leqslant \int_0^\tau \mathrm{d}\tau' |B(k,l,\tau')| \\ &= \frac{1}{(2\pi)^2}\int_0^\tau \mathrm{d}\tau' \int \mathrm{d}k' \int \mathrm{d}l' \frac{|kl'-k'l|\,|\hat{\zeta}_0(k',l')\,\hat{\zeta}_0(k-k',l-l')|}{k'^2[1+(T(\tau')-l'/k')^2]} \\ &\qquad \times \exp\{-D(k',l',\tau')-D(k-k',l-l',\tau')\}. \qquad \text{(B 21)}\end{aligned}$$

It is easily seen from (B 21) that the k', l' and τ' integrals exist and that $\hat{\zeta}(k,l,\tau)$ is absolutely integrable with respect to k and l. Thus we have shown that the nonlinear correction $\epsilon\hat{\zeta}(x,y,t)$ is uniformly bounded and remains of order ϵ for all time.

Appendix C. Wave-packet solutions

Let $\zeta^{(j)}(x,y,t)$ be the solution to (3.3) that corresponds to an initial condition $\zeta_0^{(j)}(x,y)$ with a Fourier spectrum $\hat{\zeta}_0^{(j)}(k,l)$ concentrated at the wavenumbers k_j, l_j. From (3.6) and (3.8) we have

$$\zeta^{(j)}(x,y,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \mathrm{d}k \int_{-\infty}^{\infty} \mathrm{d}l\, \hat{\zeta}^{(j)}(k,l)\, \mathrm{e}^{\mathrm{i}k\xi+\mathrm{i}ly} \exp\{Q(k,l,t)\}, \qquad \text{(C 1)}$$

where $Q = \mathrm{i}\Omega - D$, with Ω and D defined in (3.9) and (3.10). Assuming that $\hat{\zeta}_0^{(j)}(k,l)$ is small for wavenumbers outside $k_j \pm \Delta k_j$ and $l_j \pm \Delta l_j$, we can retain only the first few terms in the Taylor-series expansion of Q, namely

$$\begin{aligned}Q(k,l,t) \simeq \mathrm{i}[\Omega(k_j,l_j,t) + \Omega_k(k_j,l_j,t)\,(k-k_j) + \Omega_l(k_j,l_j,t)\,(l-l_j)] - D(k_j,l_j,t) \\ + O\left(\left(\frac{\Delta k_j}{k_j}\right)^2, \left(\frac{\Delta l_j}{l_j}\right)^2, \frac{\nu}{U'}k_j^2\left(\frac{\Delta k_j}{k_j}\right)\right), \frac{\nu}{U'}k_j^2\left(\frac{\Delta l_j}{l_j}\right). \qquad \text{(C 2)}\end{aligned}$$

Assuming that $\Delta k_j/k_j$, $\Delta l_j/l_j$ and $\nu k_j^2/U'$ are all much smaller than unity in magnitude, we drop the 'second-order' terms in the expansion of Q as indicated in (C 2). Thus (C 1) gives

$$\begin{aligned}\zeta^{(j)} &\simeq \exp[\mathrm{i}k_j\xi + \mathrm{i}l_j y + Q(k_j,l_j,t)] \\ &\quad \times \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \mathrm{d}k \int_{-\infty}^{\infty} \mathrm{d}l\, \hat{\zeta}_0^{(j)}(k,l) \exp[\mathrm{i}(k-k_j)\,(\xi+\Omega_k(k_j,l_j,t))] \\ &\qquad \times \exp[\mathrm{i}(l-l_j)\,(y+\Omega_l(k_j,l_j,t))] \\ &= \zeta_0^{(j)}(\xi+\Omega_k(k_j,l_j,t), y+\Omega_l(k_j,l_j,t)) \\ &\quad \times \exp\{\mathrm{i}[\Omega(k_j,l_j,t) - \Omega_k(k_j,l_j,t)\,k_j - \Omega_l(k_j,l_j,t)\,l_j] - D(k_j,l_j,t)\}. \qquad \text{(C 3)}\end{aligned}$$

If the initial spectrum $\hat{\zeta}_0(k,l)$ has more than one peak, the contribution from all dominant wavenumbers has to be summed, i.e.

$$\zeta(x,y,t) = \sum_{j=1}^{N} \zeta^{(j)}(x,y,t). \qquad \text{(C 4)}$$

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CORRIGENDUM

Nonlinear-wave effects on fixed and floating bodies

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Equation (2.17) requires an extra term to account for the change in location of S_f over a time step. For a point on S_f that moves vertically a distance $\Delta z (= \eta_{t+\Delta t} - \eta_t)$ in time Δt, the additional term is

$$+\tfrac{1}{2}\Delta z \left[3\left(\frac{\partial \phi}{\partial z}\right)_t - \left(\frac{\partial \phi}{\partial z}\right)_{t-\Delta t} \right].$$

This additional term has been incorporated into the calculation procedure. (A horizontal shift of the point requires a similar term as already indicated in (3.21).)